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SIMPLE WAVES ON A SHEAR FREE-BOUNDARY FLOW OF AN IDEAL INCOMPRESSIBLE FLUID

V. M. Teshukov

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The existence of simple waves is established for the system of integrodifferential equations governing the evolution of rotational free-boundary flows of an ideal incompressible fluid in a shallow-water approximation. The general properties of simple waves are analyzed. A new class of exact solutions describing the propagation of simple waves over a stationary shear flow is found.

Some exact solutions in the class of simple waves were found for this model in [1–4].

1. System of Simple-Wave Equations. The system of integrodifferential equations

$$u_t + uu_x + vu_y + gh_x = 0, \quad u_x + v_y = 0, \quad h_t + \Big(\int\limits_0^h u\,dy\Big)_x = 0 \tag{1.1}$$

describes an ideal incompressible heavy fluid flow with a free boundary $y = h(x,t)$ above an even bottom $y = 0$ in a long-wave approximation [5] ($H_0/L_0 \ll 1$, where H_0 and L_0 are the characteristic horizontal and vertical scales). Here (u, v) is the fluid-velocity vector, h is the layer depth, g is the gravitational acceleration, x and y are the Cartesian plane coordinates, and t is the time. The nonpenetration condition $v(x,0,t) = 0$ is satisfied at the layer bottom.

It is convenient to analyze the equations of motion in the Eulerian–Lagrangian coordinates x', λ, t' ($\lambda \in [0,1]$). Transformation to the new variables is given by solution of the Cauchy problem [6]

$$x = x', \quad t = t', \quad y = \Phi(x', \lambda, t'), \quad \Phi_t + u(x,\Phi,t)\Phi_x = v(x,\Phi,t), \quad \Phi(x,\lambda,0) = \Phi_0(x,\lambda) \tag{1.2}$$

[the equation $y = \Phi_0(x,\lambda)$ specifies the initial position of the Lagrangian surface $\lambda = \text{const}$; in this case, $\Phi(x,0,t) = 0$ and $\Phi(x,1,t) = h(x,t)$].

In the new coordinates Eqs. (1.1) become

$$u_t(x,\lambda,t) + u(x,\lambda,t)u_x(x,\lambda,t) + g\int\limits_0^1 H_x(x,\nu,t)\,d\nu = 0, \quad H_t(x,\lambda,t) + (u(x,\lambda,t)H(x,\lambda,t))_x = 0. \tag{1.3}$$

Here the new unknown function $H(x,\lambda,t) = \Phi_\lambda(x,\lambda,t) > 0$ is introduced. We omit the prime in the new variables. Solving this system we obtain

$$y = \int\limits_0^\lambda H(x,\nu,t)\,d\nu, \quad v(x,\nu,t) = \Phi_t(x,\nu,t) + u(x,\nu,t)\Phi_x(x,\nu,t).$$

The change of variables is reversible if $H = \Phi_\lambda \neq 0$; it is sufficient to satisfy this condition for $t = 0$.

Examples of partial solutions of the form

$$u = u(\alpha(x,t), y), \quad v = \alpha_x \tilde{v}(\alpha(x,t), y), \quad h = h(\alpha(x,t)) \tag{1.4}$$

of Eqs. (1.1) were analyzed in [1–4]. They were called the simple waves of these equations. For system (1.3), the simple waves were introduced in [7]. These solutions admit the representation

$$u = u(\alpha(x,t), \lambda), \quad H = H(\alpha(x,t), \lambda), \tag{1.5}$$

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where $\alpha(x,t)$ is a function of two variables. One can easily see that by virtue of relations (1.2), any such solution can be transformed to solution (1.4) of the initial system of equations.

In accordance with (1.3) and (1.5), the simple waves are defined by the system

$$(u(\alpha,\lambda)-k)u_\alpha(\alpha,\lambda)+g\int_0^1 H_\alpha(\alpha,\nu)d\nu=0,\ \ (u(\alpha,\lambda)-k)H_\alpha(\alpha,\lambda)+H(\alpha,\lambda)u_\alpha(\alpha,\lambda)=0,\ \ k=-\alpha_t/\alpha_x. \quad (1.6)$$

Remark. It would be more correct if solutions of the form (1.5) were called double waves, since the unknown quantities are functions of two independent variables. But since the subclass of particular solutions of system (1.3) that is characterized by the equalities $u_\lambda \equiv 0$ and $H_\lambda \equiv 0$ contains simple waves of the classical shallow-water equations, it is expedient to retain the terminology for the new class of solutions with similar properties. In this case, solutions of the form $u = u(\lambda)$ and $H = H(\lambda)$ that describe stationary shear flows [in the initial variables $u = u(y)$, $v = 0$, and $h = \text{const}$] will play the same role as stationary solutions of the shallow-water equations.

Teshukov [8] proposed extending the concept of hyperbolicity to systems of the form

$$\mathbf{U}_t(x,t,\lambda)+A\langle\mathbf{U}_x(x,t,\nu)\rangle=0. \quad (1.7)$$

Here A is the nonlocal operator acting on functions of the variable ν for any fixed x and t. In the general case, A depends on x, t, λ, and $\mathbf{U}$.

In the hyperbolic case, system (1.7) can be transformed to the characteristic form. The characteristic curves are defined by the differential equations $dx/dt = k^\beta(x,t)$:

$$\left(\mathbf{F}^\beta,\mathbf{U}_t+k^\beta\mathbf{U}_x\right)=0. \quad (1.8)$$

Here $\mathbf{F}^\beta$ are eigenfunctionals that act on functions of the variable λ and are solutions of the eigenvalue problem

$$(\mathbf{F}^\beta,A\langle\varphi\rangle)=k^\beta(\mathbf{F}^\beta,\varphi); \quad (1.9)$$

k^β are the corresponding characteristic eigenvalues and $(\mathbf{F},\varphi)$ is the result of action of the functional on the smooth trial function φ. The hyperbolicity conditions imply that all k^β satisfying (1.9) are real and the set of equalities $(\mathbf{F}^\beta,\mathbf{S})=0$ is equivalent to $\mathbf{S}=0$ in a certain class of smooth vector functions $\mathbf{S}$. In this case, Eqs. (1.7) and (1.8) are equivalent, since (1.8) is obtained by the action of $\mathbf{F}^\beta$ on (1.7).

The simple waves of Eqs. (1.7) satisfy the system

$$A\langle\mathbf{U}_\alpha\rangle-k\mathbf{U}_\alpha=0,\quad k=-\alpha_t/\alpha_x.$$

For simple waves, the characteristic relations become

$$(\mathbf{F}^\beta,(k^\beta-k)\mathbf{U}_\alpha)=(k^\beta-k)(\mathbf{F}^\beta,\mathbf{U}_\alpha)=0.$$

If $(\mathbf{F}^\beta,\mathbf{U}_\alpha)=0$ for all β, then, as follows from the aforesaid, $\mathbf{U}_\alpha=0$, i.e., $\mathbf{U}=\mathbf{U}(\lambda)$. Consequently, a nontrivial solution of the simple-wave type exists only if for each pair of values of x and t, there is an eigenvalue k^β such that $k^\beta-k=k^\beta+\alpha_t/\alpha_x=0$.

System (1.3) has the form of (1.7), if we set $\mathbf{U}=(u,H)^t$ [$(\ ,\ )^t$ denotes transposition] and

$$A\langle(f_1,f_2)^t\rangle=\left(uf_1+g\int_0^1 f_2\,d\nu,Hf_1+uf_2\right)^t.$$

It was shown in [8, 9] that for any smooth solution, there are only two real characteristic values k_1 and k_2 that satisfy the characteristic equation

$$1=g\int_0^1\frac{H\,d\nu}{(u-k)^2} \quad (1.10)$$

$[k_1 < \min_\lambda u(x,\lambda,t)$ and $k_2 > \max_\lambda u(x,\lambda,t)]$ and correspond to the discrete spectrum of the operator A. In addition, a continuous spectrum of characteristic values $k^\lambda = u(x,\lambda,t)$, where $\lambda \in (0,1)$ is determined.

Thus, we can conclude that the following possibilities for simple-wave solutions are realized: either $k = k_1(\alpha)$, or $k = k_2(\alpha)$, or $k = u(\alpha, \lambda(\alpha))$, where $\lambda(\alpha)$ is a smooth function.

In the present paper, simple waves corresponding to the discrete characteristic spectrum are studied. For determinacy, the case $k = k_2(\alpha)$ is considered [the case $k = k_1(\alpha)$ is analyzed in a similar way].

It was shown in [8] that the characteristic conditions for system (1.3) allow one to introduce the Riemann invariants:

$$R_t + uR_x = 0, \quad \omega_t + u\omega_x = 0, \qquad r_{it} + k_i r_{ix} = 0,$$
$$R = u - g\int_0^1 \frac{H'd\nu}{u' - u}, \quad \omega = u_\lambda/H, \quad r_i = k_i - g\int_0^1 \frac{H'd\nu}{u' - k_i}. \tag{1.11}$$

Here $u = u(x,\lambda,t)$, $u' = u(x,\nu,t)$, $H' = H(x,\nu,t)$, and the principal Cauchy value of the integral is used in the representation of the function R. For the simple wave in question, relations (1.11) lead to the equalities

$$R = R_0(\lambda), \quad \omega = \omega_0(\lambda), \quad r_1 = r_1^0 = \text{const}. \tag{1.12}$$

The functions $R_0(\lambda)$ and $\omega_0(\lambda)$ and the constant r_1^0 are determined from the conditions of continuous joining of a simple wave to the given shear flow $u = u_0(\lambda)$, $H = H_0(\lambda)$ along the boundary characteristics $\alpha = \text{const}$. The function $k(x,t) = k_2(x,t)$ is governed by the equation

$$k_t + kk_x = 0. \tag{1.13}$$

Any solution of Eq. (1.13) and relations (1.12) define a simple wave $u = u(k,\lambda), H = H(k,\lambda)$. The functions u and H are found by solution of the system of nonlinear integral equations following from (1.11) and (1.12). In view of the complexity of these equations, we give a direct proof of the existence of a solution of system (1.6), using relations (1.12) as integrals of this system.

2. Existence and Properties of Simple Waves. Below, as a simple wave parameter α, we consider the depth

$$h(x,t) = \int_0^1 H\, d\nu.$$

The characteristics of a simple wave move with constant velocities $dx/dt = k$, and the domain of definition of a simple wave in the space x, λ, t is covered by a one-parameter family of planes $[h(x,t) = \text{const}]$. In each plane, the functions u and H depend only on λ. It is natural to consider the problem of continuous joining of a simple wave to the given shear flow at the characteristics $h = h_m = \text{const}$:

$$u_h = -g(u-k)^{-1}, \quad H_h = gH(u-k)^{-2}, \quad h = \int_0^1 H d\nu, \quad u\Big|_{h=h_m} = V_0(\lambda), \quad H\Big|_{h=h_m} = H_0(\lambda). \tag{2.1}$$

Here h_m is the constant depth of the adjoining shear flow. Here and below, the larger characteristic root k_2 of Eq. (1.10) is denoted by k.

From (2.1) follows the property: if $u(h_m,\lambda_1) = u(h_m,\lambda_2)$, then $u(h,\lambda_1) = u(h,\lambda_2)$ everywhere in the simple-wave domain. This is obvious, because the equation for the difference $\delta = u(h,\lambda_1) - u(h,\lambda_2)$ is homogeneous:

$$\delta_h = g\delta(u(h,\lambda_1) - k)^{-1}(u(h,\lambda_2) - k)^{-1}, \quad \delta(h_m) = 0.$$

Therefore, if the horizontal velocity is a monotonic function of λ for $h = h_m$ $[u_0'(\lambda) \neq 0]$, then $u_\lambda(h,\lambda) \neq 0$.

The problem of continuous joining of a simple wave to the given shear flow with a nonmonotonic velocity profile can be reduced to the problem with a monotonic profile. Indeed, consider the velocity profile $V_0(\lambda)$ shown in Fig. 1: $V_0'(\lambda) > 0$ for $0 \leqslant \lambda < \lambda_*$ and $V_0'(\lambda) < 0$ for $\lambda_* < \lambda \leqslant 1$. Let $V_0(\lambda_1) = V_0(1)$. We define the function $f(\lambda)$ in the interval (λ_1, λ_*) by the equality $V_0(\lambda) = V_0(f(\lambda))$. According to the above properties, the equality $u(h,\lambda) = u(h, f(\lambda))$ $[\lambda_* \leqslant f(\lambda) \leqslant 1]$ is valid in the simple-wave domain. We introduce the function $H_1(h,\lambda) = H(h, f(\lambda))f'(\lambda)$ and the function $H_2(h,\lambda)$ which is defined as follows: $H_2(h,\lambda) = H(h,\lambda)$ for $\lambda \in [0,\lambda_1]$, and $H_2(h,\lambda) = H(h,\lambda) - H_1(h,\lambda)$ for $\lambda \in (\lambda_1, \lambda_*)$. Since

$$\int_0^1 \frac{H d\nu}{(u-k)^s} = \int_0^{\lambda_1} \frac{H(h,\nu)d\nu}{(u(h,\nu)-k)^s} + \int_{\lambda_1}^{\lambda_*} \frac{(H(h,\nu) - H(h,f(\nu))f'(\nu))d\nu}{(u(h,\nu)-k)^s} = \int_0^{\lambda_*} \frac{H_2(h,\nu)d\nu}{(u(h,\nu)-k)^s},$$
$$s = 0,\ 1,\ 2, \ldots,$$

a problem of the form (2.1) for $u(h,\lambda)$ and $H_2(h,\lambda)$ in the interval $\lambda \in [0,\lambda_*]$ with a monotonic velocity profile is obtained: $V_0'(\lambda) \neq 0$ for $\lambda \in [0,\lambda_*)$ (the integrals over the interval [0, 1] in (1.10) and (2.1) should be replaced by the integrals over the interval $[0,\lambda_*]$). If this problem is solved, one should solve additionally the linear problem $K_h = gK(u-k)^{-2}$ and $K(h_m,\lambda) = H_0(\lambda) + H_0(f(\lambda))f'(\lambda)$ for the unknown function $K(h,\lambda) = H(h,\lambda) + H_1(h,\lambda)$.

Next, we can find a solution of the initial problem with a nonmonotonic velocity profile: for $\lambda \in [0,\lambda_1]$, it coincides with the solution of the reduced problem; $u(h,\lambda)$ coincides with the solution of the reduced problem and $H = 2^{-1}(H_2 + K)$ for $\lambda \in [\lambda_1, \lambda_*]$; $u(h,\lambda) = u_r(h, f^{-1}(\lambda))$ and $H(h,\lambda) = [(2f')^{-1}(K - H_2)](h, f^{-1}(\lambda))$ for $\lambda \in [\lambda_*, 1]$. Here u_r is a solution of the reduced problem and $f^{-1}(\lambda)$ is a function that is inverse to f. The case of a nonmonotonic velocity profile with a finite number of points at which the derivative changes sign is considered in a similar way. It should be noted that the reduced problem has the following peculiarities: first. the functions $H_2(h_m,\lambda)$ and $H_2(h,\lambda)$ are discontinuous for $\lambda = \lambda_1$; second, $u_\lambda(h,\lambda_*) = 0$.

In what follows, we assume that $V_0'(\lambda) \neq 0$ if $\lambda \neq 1$. We shall consider the case of $V_0' \geqslant 0$ (the problem in the interval $[0,\lambda_*]$ reduces to the problem in the interval [0,1] by dilatation of the variable λ).

Differentiating the characteristic equation yields the following differential equation for the function $k(h)$:

$$k_h = -\frac{3}{2}\int_0^1 \frac{H d\nu}{(u-k)^4}\Big(\int_0^1 \frac{H d\nu}{(u-k)^3}\Big)^{-1}, \quad k(h_m) = k_2^0. \tag{2.2}$$

The initial condition for this equation is the coincidence of the characteristic velocity on the boundary with the larger real root k_2^0 of Eq. (1.10) for $u = V_0(\lambda)$ and $H = H_0(\lambda)$. From (2.1) one can also obtain the following differential equation for the function $u_\lambda(h,\lambda)$:

$$u_{\lambda h} = g u_\lambda (u-k)^{-2}, \quad u_\lambda(h_m,\lambda) = V_0'(\lambda). \tag{2.3}$$

Let us prove the local existence of a solution of problem (2.1)–(2.3) assuming that $V_0(\lambda)$ is a continuously differentiable function, $H_0(\lambda)$ is a continuous function, $H_0(\lambda) > \delta > 0$, and $k_0 - V_0(\lambda) > \delta > 0$. We introduce a Banach space B of elements $\mathbf{V} = (u, u_\lambda, H, k)$ with a norm $\|\mathbf{V}\| = \max\limits_\lambda |u| + \max\limits_\lambda |u_\lambda| + \max\limits_\lambda |H| + |k|$.

The first three components of the vector $\mathbf{V}$ are continuous functions of the variable $\lambda \in [0,1]$ and the last component is a real number. Problem (2.1)–(2.3) is representable in the general form

$$d\mathbf{V}/dh = \mathbf{F}(\mathbf{V}), \quad \mathbf{V}(h_m) = \mathbf{V}_0. \tag{2.4}$$

Here $\mathbf{F}(\mathbf{V})$ is a nonlinear operator in the space B. Following the known existence theorem for differential equations in a Banach space [10], if there is $\varepsilon > 0$ such that the inequalities

$$\|\mathbf{F}(\mathbf{V})\| \leqslant M, \quad \|\mathbf{F}(\mathbf{V}_1) - \mathbf{F}(\mathbf{V}_2)\| \leqslant K\|\mathbf{V}_1 - \mathbf{V}_2\| \tag{2.5}$$

are valid for $\|\mathbf{V} - \mathbf{V}_0\| < \varepsilon$, problem (2.4) has a unique solution $\mathbf{V}(h) \in B$ for $|h - h_m| < \delta_1 = \min(\varepsilon M^{-1}, K^{-1})$, such that $\|\mathbf{V} - \mathbf{V}_0\| < \varepsilon$.

Let us verify that the conditions of the theorem are satisfied. We consider a ball $\|\mathbf{V} - \mathbf{V}_0\| < 2^{-1}\delta$. For elements of this ball, we have $|u-k| > 2^{-1}\delta$ and $|H| > 2^{-1}\delta$. Indeed,

$$|u - k| = |V_0 - k_0 + u - V_0 + k_0 - k| \geqslant |V_0 - k_0| - \|\mathbf{V} - \mathbf{V}_0\| > 2^{-1}\delta,$$
$$|H| = |H_0 + H - H_0| \geqslant |H_0| - \|\mathbf{V} - \mathbf{V}_0\| > 2^{-1}\delta.$$

By virtue of the continuity of mappings that define $\mathbf{F}(\mathbf{V})$ on the set $|u-k| > 2^{-1}\delta$ and $|H| > 2^{-1}\delta$, there are constants $M(\delta, \|\mathbf{V}_0\|)$ and $K(\delta, \|\mathbf{V}_0\|)$ for which inequalities (2.5) are valid. Then, following the above theorem, the solution of problem (2.4) exists and it is unique for $|h-h_m| < \delta_1(\delta, \|\mathbf{V}_0\|)$. As mentioned above, the problem with a nonmonotonic velocity profile $V_0(\lambda)$ requires introducing discontinuities of the function $H(h,\lambda)$. The local existence theorem for a solution can be obtained in this case as well. If $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < 1$ are the points of discontinuity of the function H, as elements of B we consider vectors $(u_1, u_{1\lambda}, H_1, u_2, u_{2\lambda}, H_2, \ldots, u_{n+1}, u_{n+1\lambda}, H_{n+1}, k)$, whose first components are continuous functions in the intervals $[\lambda_j, \lambda_{j+1}]$ ($\lambda_0 = 0$ and $\lambda_{n+1} = 1$) and all arguments are repeated. For the solution obtained, one can prove, using Eqs. (2.1)–(2.3), that u is a continuously differentiable function with respect to λ, while H is a piecewise continuous function with points of discontinuity $\lambda_1, \ldots, \lambda_n$.

Let us prove the global existence theorem with respect to h for the simple-wave problem, assuming that

$$V_0'(\lambda) > 0,\ \ \lambda \in [0,1),\ \ k_0 > V_0(1) + \delta,\ \ (\omega_0(\lambda))^{-1} = H_0(\lambda)(V_0'(\lambda))^{-1} \geqslant a > 0,\ \ H_0(\lambda) > \delta > 0. \tag{2.6}$$

Lemma. *For each solution of problem* (2.1)–(2.3), *the following inequalities are satisfied:*

$$(gh)^{1/2} + a^{-1}h \geqslant k - u \geqslant (gh)^{1/2}(1 + h^{1/2}g^{-1/2}a^{-1})^{-1}. \tag{2.7}$$

Proof. In a neighborhood of $h = h_m$, the inequalities $k > u_1 > u > u_0$ [$u_1 = u(h,1)$ and $u_0 = u(h,0)$] are satisfied by virtue of continuity. As mentioned above, the relations

$$g\int_0^1 \frac{H'd\nu}{(u'-k)^2} = g\int_{u_0}^{u_1} \frac{1}{\omega'}\frac{du'}{(u'-k)^2} = 1, \quad \omega = \omega_0(\lambda) = V_0'(\lambda)(H_0(\lambda))^{-1} \tag{2.8}$$

give integrals of Eqs. (2.1)–(2.3). Using the inequalities

$$0 \leqslant k - u_1 \leqslant k - u \leqslant k - u_0, \tag{2.9}$$

from (2.8), we obtain

$$(k-u_0)^2 \geqslant g\int_0^1 H d\nu = gh, \quad (k-u_1)^2 \leqslant g\int_0^1 H d\nu = gh.$$

The relation

$$h = \int_0^1 H d\nu = \int_{u_0}^{u_1} \omega^{-1} du$$

and the inequality $\omega^{-1} \geqslant a$ give $h \geqslant a(u_1 - u_0)$.

From (2.8), we have

$$1 = g\int_{u_0}^{u_1} \frac{1}{\omega'}\frac{du'}{(u'-k)^2} \geqslant ga\left[\frac{1}{u_0-k} - \frac{1}{u_1-k}\right],$$

and, hence, $k-u_1 \geqslant ga(k-u_0)[k-u_0+ga]^{-1} \geqslant (gh)^{1/2}(1+h^{1/2}g^{-1/2}a^{-1})^{-1}$. Since $k-u_0 = k-u_1+u_1-u_0 \leqslant (gh)^{1/2} + a^{-1}h$, the inequalities of the lemma follow from (2.9).

Using *a priori* estimates (2.7) and system (2.1)–(2.3), we can prove that $u(h,\lambda)$, $u_\lambda(h,\lambda)$, $H(h,\lambda)$, and $k(h)$ are bounded in any interval $h \in [\sigma, A]$ ($0 < \sigma < h_m < A$). The inequalities $|u-k| > \varepsilon(\sigma, A, \|\mathbf{V}_0\|)$ and $H > \varepsilon(\sigma, A, \|\mathbf{V}_0\|)$ hold in this interval.

Sequential application of the local existence theorem gives the existence of a solution of problem (2.1)–(2.3) over the entire interval $h \in (0, A)$. Thus, we proved

Theorem. *Let the function $V_0(\lambda)$ be continuously differentiable in the interval* [0, 1] *and the function $H_0(\lambda)$ piecewise continuous with a finite number of points of discontinuity of the first kind. Let conditions* (2.6) *be satisfied. Then problem* (2.1)–(2.3) *has a unique solution in any interval $h \in (0, A]$ ($h_m \in (0, A]$), with the continuously differentiable function $u(\lambda, h)$ and the piece-wise continuous function $H(\lambda, h)$.*

As was mentioned above, the monotony of the velocity profile does not lead to the loss of generality. and the condition $(\omega_0(\lambda))^{-1} \geqslant a$ implies boundedness of the derivative u_y on the boundary characteristic $h = h_m$ ($\omega = u_y$ in accordance with the relations between x, y, t and x, λ, t).

If $h \to 0$ on the closing simple-wave characteristics, the solution describes the spreading of the shear flow over a dry bed. It follows from (2.7) that $u \to k$ as $h \to 0$, which implies smoothing of the flow velocity over the depth. The flow velocity over a dry bed is defined by the equality $u = k = r_1^0$. Indeed, the relation

$$r_1 = k_1 - \int_0^1 \frac{H d\nu}{(u - k_1)} = r_1^0 = \text{const}$$

gives an integral of Eqs. (2.1)–(2.3). Using the Cauchy inequality, we obtain

$$\left| \int_0^1 \frac{H d\nu}{(u-k_1)} \right| \leqslant \Big(\int_0^1 \frac{H d\nu}{(u-k_1)^2} \Big)^{1/2} \Big(\int_0^1 H d\nu \Big)^{1/2} = h^{1/2}, \quad 1 = g \int_0^1 \frac{H d\nu}{(u-k_1)^2} \leqslant \frac{1}{(u_1 - k_1)^2} gh.$$

Therefore, $u_1 \to k_1$ and $k_1 \to r_1 = r_1^0$ as $h \to 0$.

In the simple wave considered, the flow velocity $u(h, \lambda)$ increases along each Lagrangian surface $\lambda =$ const, if the level h decreases. The wave increasing the fluid level h decreases $u(h, \lambda)$. Since $k > u$, the fluid particles enter the simple-wave zone from the right (with respect to the x direction). For a left simple wave ($k < u$), the behavior of $u(h, \lambda)$ changes to the opposite. If $V_0(\lambda) > 0$ in the upstream flow, the passage of a right simple wave does not produce a critical layer ($u = 0$) in the stationary downstream shear flow. If $V_0(\lambda) < 0$, the critical layer can be produced by the passage of the wave.

To complete the construction of the simple wave, we have to solve the Cauchy problem $h_t + k(h)h_x = 0$ and $h(x, 0) = h_0(x)$. Any solution of this equation determines a pair of functions $u(x, \lambda, t) = u(h(x, t), \lambda)$ and $H(x, \lambda, t) = H(h(x, t), \lambda)$ that satisfy system (1.3). If $h_0'(x) > 0$, the solution gives a right wave decreasing the fluid level. Since $k'(h) > 0$, the Cauchy problem has a smooth solution for $t > 0$. If there are points x such that $h_0'(x) < 0$ for $t = 0$, zones with increasing fluid levels appear and, as is known, breaking of the wave will happen: the derivatives of the function h become unbounded at a certain moment of time. Further description of the flow evolution requires constructing discontinuous functions [11].

3. Exact Solution of the Type of a Simple Wave. As was mentioned above, the Riemann invariants R and ω^{-1} depend only on λ, and, hence, they are functionally dependent in a simple wave.

We consider the simple case $R = B\omega^{-1}$, where $B =$ const. It is convenient to represent B in the form $B = g\pi\text{cotan}\mu\pi$, where μ is a real parameter. We determine the function ω^{-1} from the integral equation

$$u - g \int_{u_0}^{u_1} \frac{1}{\omega'} \frac{du'}{u' - u} = \frac{1}{\omega} g\pi \,\text{cotan}\mu\pi. \tag{3.1}$$

Here, assuming that $u_\lambda \neq 0$ (and, hence, $\omega^{-1} \neq 0$) for $0 < \lambda < 1$, we choose u as the integration variable in the formula representing the Riemann invariant R. Equation (3.1) is a linear singular integral equation for the unknown ω^{-1}. According to the general theory of singular integral equations [12], Eq. (3.1) has a unique solution in the class of functions bounded at the point $u = u_0$ and unbounded at the point $u = u_1$. Using the methods developed in [12], we obtain a solution in the form (assuming that $0 < \mu < 1$)

$$\omega^{-1} = \sin \mu\pi (g\pi)^{-1} (u - \mu(u_1 - u_0)) \Big(\frac{u - u_0}{u_1 - u} \Big)^{\mu}. \tag{3.2}$$

Using formulas (1.11) we find the Riemann invariants r_i:

$$r_i = (k_i - \mu(u_1 - u_0)) \Big(\frac{u_0 - k_i}{u_1 - k_i} \Big)^{\mu} \qquad (i = 1, 2). \tag{3.3}$$

Here the characteristic velocity k_i satisfies Eq. (1.10) or the equivalent equation $\partial r / \partial k = 0$ with the function $r(k, u_1, u_0)$ defined by (3.3). The velocities k_i are roots of the quadratic equation

$$k^2 - (u_1 + u_0 + \mu(u_1 - u_0))k + u_1 u_0 + \mu^2 (u_1 - u_0)^2 = 0. \tag{3.4}$$

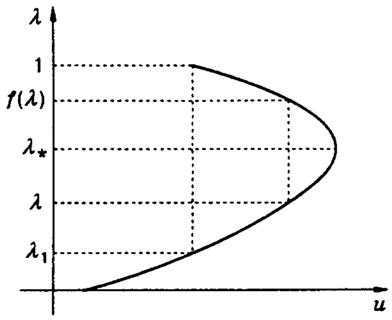

Fig. 1

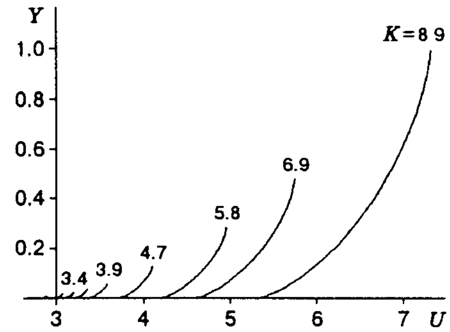

Fig. 2

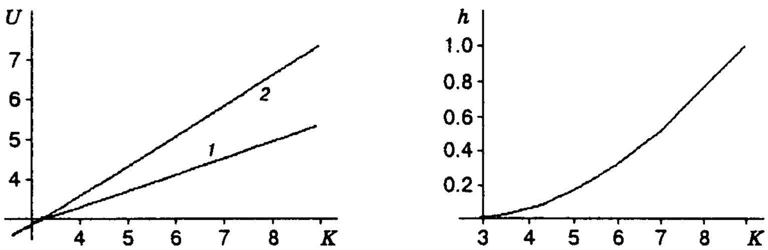

Fig. 3

A simple wave is defined if we set

$$\omega^{-1} = f(\lambda), \quad r_1 = r_1^0, \quad k_2 = k(x,t), \tag{3.5}$$

where $k(x,t)$ is any solution of Eq. (1.13). In accordance with the assumptions used in constructing the solution (3.2), we have $f(\lambda) > 0$ for $\lambda \in (0,1)$, and $f(0) = 0$ and $f(\lambda) \to \infty$ for $\lambda \to 1$. With this choice, the function $u(k,\lambda)$ determined from (3.5) satisfies the inequalities

$$0 < u_0 \leqslant u \leqslant u_1 \leqslant \mu^{-1}(1+\mu)u_0. \tag{3.6}$$

We introduce the dimensionless quantities $K_i = 2k_i(u_{1m} - u_{0m})^{-1}$, $U_i = 2u_i(u_{1m} - u_{0m})^{-1}$, and $R_i = 2r_i(u_{1m} - u_{0m})^{-1}$, where u_{1m} and u_{0m} are the values of u_1 and u_0 for $h = h_m$.

Equation (3.4) has the real roots

$$K_{1,2} = z\left[\gamma + \mu \mp \sqrt{1+2\gamma\mu}\right], \quad z = \frac{u_1 - u_0}{u_{1m} - u_{0m}}, \quad \gamma = \frac{u_1 + u_0}{u_1 - u_0}, \tag{3.7}$$

since $\gamma > (1+2\mu)$ by virtue of (3.6). The relations

$$R_1 = (\gamma_m - \mu - \sqrt{1+2\gamma_m\mu})\left(\frac{\sqrt{1+2\gamma_m\mu} - 1 - \mu}{\sqrt{1+2\gamma_m\mu} + 1 - \mu}\right)^{\mu},$$

$$z = R_1(\gamma - \mu - \sqrt{1+2\gamma\mu})^{-1}\left(\frac{\sqrt{1+2\gamma\mu} + 1 - \mu}{\sqrt{1+2\gamma\mu} - 1 - \mu}\right)^{\mu},$$

$$K = K_2 = z(\gamma + \mu + \sqrt{1+2\gamma\mu}), \qquad U_1 = z(1+\gamma), \quad U_0 = z(\gamma - 1)$$

define the dependences $U_1(K)$, and $U_0(K)$ in parametric form, with the parameter γ varying in the interval $(1+2\mu, \infty)$; γ_m is the fixed value from this interval corresponding to the boundary characteristics $h = h_m$. The function $H(k,\lambda)$ is given by the relation $H = u_\lambda \omega^{-1}$. Transformation to the variable y is defined by the formula ($y = h_m Y$)

$$Y = (h_m)^{-1} \int_{u_0}^{u} \omega^{-1} du = \frac{\sin \mu\pi}{\mu\pi} \frac{z^2}{\gamma_m - \mu} [(\gamma + 1 - 2\mu) B_s(1+\mu, 1-\mu) - 2B_s(1+\mu, 2-\mu)],$$

where $B_s(p,q)$ is an incomplete β-function and $s = (u - u_0)(u_1 - u_0)^{-1}$. The fluid-layer depth is expressed in the form $h = h_m z^2(\gamma - \mu)(\gamma_m - \mu)^{-1}$. The parameter γ_m is related to the Froude number $\mathrm{Fr}_m = (u_{1m} - u_{0m})(gh_m)^{-1/2}$ by the relation $\mathrm{Fr}_m = \sqrt{2}(\mu(\gamma_m - \mu))^{-1/2}$.

Figure 2 show velocity profiles for $\mu = 1/3$ and $\mathrm{Fr}_m = 1$ with variation of K in the simple wave describing the spreading of the shear flow over a dry bed [$U = 2u(u_{1m} - u_{0m})^{-1}$]. Each curve is a graph of U for fixed K. Figure 3 shows graphs of the bottom and the free-surface velocities (curves 1 and 2) and a graph of the function $h_m^{-1}h(K)$. One can see that the flow velocity is smoothed as $h \to 0$ and tends to the value of the Riemann invariant r_1^0. The resulting exact solution is expressed as well as the Freeman solution in terms of incomplete β-functions but describes another flow.

As a result, we established the existence of simple-wave solutions for system (1.3). It should be noted that the sufficient conditions for hyperbolicity of system (1.3) obtained in [9] can be formulated only in terms of the Riemann invariants R and ω in the case of an incompressible fluid (ρ = const). Owing to the conservation of these invariants in a simple wave, this solution belongs to the region of hyperbolicity of equations if these equations are hyperbolic for the shear flow adjoining to the simple wave. The conservation of Riemann invariants in the simple wave determines situations in which such a flow appears. If a perturbation front (discrete-spectrum characteristic) moves over an undisturbed shear flow, the flow in a region behind these characteristics is a simple wave. This follows from the fact that three of the four Riemann invariants in the region behind the front are the same as in the incident shear flow.

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